CS 4641 Background Test

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1 Probability and Statistics 1

1.1 What are the median and mean of a set $\{15, 8, 3, 7, 12\}$, respectively?

A. $9, 15$ B. $15, 9$ C. $8, 9$ D. $9, 8$

1.2 Assume X is a random variable with the variance of $Var[X]$. What would be the variance of $aX + b + c$ where a, b, c are scalar constants?

1.3 Let X_1, $X_2, \ldots, X_n$ be a set of samples drawn independently and identically from a distribution $P(X)$ over a one dimensional random variable $X \in \mathbb{R}$. What is the formula to estimate the variance?

1.4 Suppose that given random variable X, random variables Y and Z are conditionally independent. Which one of the following is true?

A. $P(Y|X, Z) = P(Y|Z)$ B. $P(Y|X) = P(Z|X)$ C. $P(Y|Z, X) = P(Y|X)$

1.5 Assume that the cumulative distribution function (C.D.F.) of a random variable is given as follows. Write down the expression for the probability density function (p.d.f.) of X.

$$F(x) = \begin{cases} 0 & if \quad x < 0 \\ x & if \quad 0 \leq x < 1 \\ 1 & if \quad x \geq 1 \end{cases}$$

1.6 One box has several different colored balls inside of it. It has 1 red, 2 green, 4 blue, and 8 yellow balls. All the balls are the same size and shape. If you take out a ball from the box without looking, what is the probability that it will NOT be a yellow ball.

A. $\frac{8}{15}$ B. $\frac{7}{8}$ C. $\frac{7}{15}$ D. $\frac{6}{15}$

1.7 Which of the following statements is true? (Notation $A \cup B$ means "either event A or B happens"; notation $A \cap B$ means "both event A and B happened simultaneously").

A. If the events A and B are independent, then $P(A \cup B) = P(A) + P(B)$
B. $P(A \cup B) < P(A) + P(B)$
C. If A and B are mutually exclusive, then $P(A \cap B) = P(A)P(B)$
D. None of the above.

1.8 Suppose we throw two dice (each with 6 sides) simultaneously. Assuming the two dices are independent and each side comes up with equal probability. What is the probability of getting sum as 8 of the two numbers shown?

1.9 Suppose that $f(x)$ is the probability density function (p.d.f.) of random variable X. Then, is $f(x|b < x \leq a) = \frac{f(x)}{F(a)-F(b)}$ for any $b < x \leq a$ True or False?

1.10 Assume that we have m coins. We toss each one of them n times. The probability of heads showing up for each coin is p. What's the probability of getting n heads for some coin? Your answer should be in terms of m, n and p.

2 Probability and Statistics 2

2.1 Suppose that there are two events A and B, and furthermore one event is included in another event: $A \subseteq B$ such that $P(A) > 0$ and $P(B) > 0$. How many of the following statements are true: **(1)** $P(B|A) = 1$, **(2)** $P(A|B) = P(A)/P(B)$?

A. 0 B. 1 C. 2

2.2 Let $Y_1, Y_2, \ldots, Y_n$ be a set of random sample of size n drawn independently and identically from a normal distribution with mean μ and variance σ^2. Then let $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. How many of the following statements are true: **(1)** $\overline{Y}$ is normally distributed, **(2)** mean $\mu_{\overline{Y}} = \mu/n$, and **(3)** variance $\sigma^2_{\overline{Y}} = \sigma^2/n$.

A. 0 B. 1 C. 2 D. 3

2.3 Is the following statement TRUE or FALSE: If we perform the same experiment a large number of times and the results are independent and identically distributed, then the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed.

2.4 Suppose random variables Y, x and ϵ are related by $Y = \beta_0 + \beta_1 x + \epsilon$, with β_0 and β_1 are parameters and ϵ is assumed to independent of x and follow normal distribution with mean 0 and constant variance. Please calculate: **(1)** $E(\epsilon|x)$, and **(2)** $E(Y|x)$.

2.5 Is the following statement TRUE or FALSE: $p\left(X = x|Y = y\right) = \frac{p(X=x)p(Y=y|X=x)}{\sum'_x p(X=x')p(Y=y|X=x')}$

2.6 The population correlation coefficient $\rho_{X,Y}$ between two random variables X and Y with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as: $\rho_{X,Y} = \mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$. How many of the following statements are true: **(1)** X, Y independent $\Rightarrow \rho_{X,Y} = 0$ **(2)** $\mathrm{corr}(X,Y) \neq \mathrm{corr}(Y,X)$

A. 0 B. 1 C. 2

2.7 Is the following statement TRUE or FALSE: If X and Y are ***any*** two random variables and a and b are ***any*** two constants, then $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$?

2.8 Suppose Y_1 and Y_2 are any two random variables, the ***conditional expectation*** of $g(Y_1)$, given that $Y_2 = y_2$, is defined to be: **(1)** $E\left(g(Y_1)|Y_2 = y_2\right) = \int_{-\infty}^{+\infty} g(y_1) f(y_1|y_2)\, dy_1$, or **(2)** $E\left(g(Y_1)|Y_2 = y_2\right) = \int_{-\infty}^{+\infty} g(y_1) f(y_1|y_2)\, dy_2$

A. (1) B. (2)

2.9 Suppose $F(y)$ is a cumulative distribution function over real domain, then how many of the following statements are TRUE: **(1)** $F(-\infty) = 0$, **(2)** $F(+\infty) = 0$, and **(3)** If y_1 and y_2 are any values such that $y_1 < y_2$, then $F(y_1) \leq F(y_2)$?

A. 0 B. 1 C. 2 D. 3

2.10 **Suppose $X \sim Normal\,(5, 3^2)$. What is $P\,(X \leq 8)$ in terms of the standard normal variable Z? (Or $P(X \leq 8)$ is equal to?)**

A. $P\,(Z \leq 1)$ B. $P\,(Z \leq -1)$ C. $P\,(Z \leq 0.6)$

3 Linear Algebra

3.1 **We have two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times d}$. Let A_{ij} be the entry of A in the i-th row and j-th column, and similar notation applies for B_{ij}. Let the multiplication of A and B be $C = AB$, what is the dimensions of matrix C and how do you compute entry C_{ij}?**

3.2 **If we have $a + 2b - 3c = 0$, where $a, b, c \in \mathbb{R}^n$ are vectors, then these vectors are linearly independent. True or False?**

3.3 **The solution for the linear system $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ and $m \neq n$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ is $x = A^{-1}b$. True or False?**

3.4 **If x is in the null space of A, then**

A. $Ax = 0$ B. $Ax = x$ C. $x^T A = 0$ D. $x^T A = x^T$

3.5 **If Q is an orthonormal matrix, which of the following statements are correct? Choose all that apply**

A. $QQ = I$ B. $QQ^T = I$ C. $Q^T Q = I$ D. $Q = Q^{-1}$ E. $Q^T = Q^{-1}$

3.6 **The squared Frobenius norm $\|X\|_F^2$ can also be expressed as**

A. $\mathtt{tr}(X^T X)$ B. $\mathtt{det}(X^2)$ C. $XX^{-1}X$

3.7 **What is the rank and trace of the matrix A**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 6 & -2 \end{pmatrix}$$

3.8 **Which vectors are in the column space of the matrix A in Question 3.7? Choose all that apply.**

A. $[1, 2, 0]^T$ B. $[1, 0, -2]^T$ C. $[0, 0, 0]^T$ D. $[1, 3, 4]^T$ E. $[0, 2, 0]^T$

3.9 **Suppose $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix, and it has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and the corresponding n orthonormal eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}$, which one of the following is true? (choose one)**

A. $Av_1 = \lambda_1 v_1$ B. $A = \sum_{i=1}^n \lambda_i v_i v_i^\top$ C. Both A and B. D. None of the above.

3.10 **Let $f(x) = a^\top x$, and $g(x) = (a^\top x)^2$ where $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$. Write the derivatives of $f(x)$ and $g(x)$ with respect to x in matrix and vector format, i.e., $\nabla f(x)$ and $\nabla g(x)$.**

4 Numpy: Answer with numpy commands.

Assume that the following import statement is available
(from numpy import *
from math import *)

4.1 **Create a 1D array of all the even numbers between 31 and 75.**

4.2 **Let x = array([2, 5, 1, 6]). Compute the square root of each element, and assign the result to a.**

4.3 **Let x = array([3, 2, 6, 8]), y = array([4, 1, 3, 5]). Answer in Numpy commands. Raise each element of x to the power specified by the corresponding element in y.**

4.4 **Given the array A = array([[2, 4, 1], [6, 7, 2], [3, 5, 9]]), assign the first row of A to a numpy array called a.**

4.5 **Given x = array([3, 15, 9, 12, -1, 0, -12, 9, 6, 1]), set the values of x that are positive to zero.**

4.6 **Given the following code, calculate the result of w:**

```
if z < 5:
    w = 2 * z
elif z < 10:
    w = 9 - z
elif z < 100:
    w = sqrt(z)
else:
    w = z
```

1. $z = 1$, $w = ?$
   A. 2
   B. 8
   C. sqrt(1)
   D. 200
2. $z = 9$, $w = ?$
   A. 18
   B. 0
   C. sqrt(9)
   D. 200
3. $z = 60$, $w = ?$
   A. 120
   B. -51
   C. sqrt(60)
   D. 200
4. $z = 200$, $w = ?$
   A. 400
   B. -191
   C. sqrt(200)
   D. 200

4.7 **Given A = array([[1, 2, 3], [4, 5, 6]]), b = 7 and $C = A.T$ * b, what is the result of C?**

4.8 **Given A = array([[1, 2, 3], [4, 5, 6]]), B = array([[9, 8, 7], [6, 5, 4]]) and $C = A$ * B, what is the result of C?**

4.9 **Given A = array([3, 6, 2, 5, 9]) and B = array([9, 5, 3, 7, 1]), what is the resulting array returned by command $A <= B$?**

4.10 **Given the array x = array([1, 8, 3, 9, 0, 1]), add up the values of the elements of x, and assign the result to a.**